

FRACTIONAL TYPE INTEGRAL OPERATORS ON VARIABLE HARDY SPACES

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ABSTRACT. Given certain $n \times n$ invertible matrices A_1, \dots, A_m and $0 \leq \alpha < n$, in this paper we obtain the $H^{p(\cdot)}(\mathbb{R}^n) \rightarrow L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the integral operator with kernel $k(x, y) = |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m}$, where $\alpha_1 + \dots + \alpha_m = n - \alpha$ and $p(\cdot), q(\cdot)$ are exponent functions satisfying log-Hölder continuity conditions locally and at infinity related by $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$. We also obtain the $H^{p(\cdot)}(\mathbb{R}^n) \rightarrow H^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the Riesz potential operator.

1. INTRODUCTION

Given a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ such that $0 < \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) < \infty$, let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the space of measurable functions such that for some $\lambda > 0$,

$$\int \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty.$$

We set

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

We see that $(L^{p(\cdot)}(\mathbb{R}^n), \|f\|_{p(\cdot)})$ is a quasi normed space. As usual we will denote $p_+ = \sup_{x \in \mathbb{R}^n} p(x)$ and $p_- = \inf_{x \in \mathbb{R}^n} p(x)$.

These spaces are referred to as the variable L^p spaces. In the last years many authors have extended the machinery of classical harmonic analysis to these spaces. See, for example [1], [2], [4], [5], [7].

In the famous paper [6], C. Fefferman and E. Stein defined the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, with the norm given by

$$\|f\|_{H^p} = \left\| \sup_{t>0} \sup_{\varphi \in \mathcal{F}_N} |t^{-n} \varphi(t^{-1} \cdot) * f| \right\|_p,$$

for a suitable family \mathcal{F}_N . In the paper [9], E. Nakai and Y. Sawano defined the Hardy spaces with variable exponents, replacing L^p by $L^{p(\cdot)}$ in the above norm and they investigate their several properties.

Let $0 \leq \alpha < n$, let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a measurable function, and let $q(\cdot)$ be defined by $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$. Given certain invertible matrices A_1, \dots, A_m , $m \geq 1$, we

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$$(1) \quad T_\alpha f(x) = \int |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m} f(y) dy,$$

where $\alpha_1 + \dots + \alpha_m = n - \alpha$. We observe that in the case $\alpha > 0$, $m = 1$ and $A_1 = I$, T is the classical fractional integral operator (also known as the Riesz potential) I_α .

With respect to classical Lebesgue or Hardy spaces, in the case $m > 1$, in the paper [11], we obtained the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of these operators and we show that we cannot expect the $H^p(\mathbb{R}^n) - H^q(\mathbb{R}^n)$ boundedness of them. This is an important difference with the case $m = 1$. Indeed, in the paper [13], M. Taibleson and G. Weiss, using the molecular characterization of the real Hardy spaces, obtained the boundedness of I_α from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, $0 < p \leq 1$. In this paper we will extend both results to the setting of variable exponents. Here and below we shall postulate the following conditions on $p(\cdot)$,

$$(2) \quad |p(x) - p(y)| \leq \frac{c}{-\log|x - y|}, \quad |x - y| < \frac{1}{2},$$

and

$$(3) \quad |p(x) - p(y)| \leq \frac{c}{\log(e + |x|)}, \quad |y| \geq |x|.$$

We note that the condition (3) is equivalent to the existence of constants C_∞ and p_∞ such that

$$(4) \quad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

In Section 2 we recall the definition and atomic decomposition of the Hardy spaces with variable exponents given in [9]. We also state three crucial lemmas, two of them referring to estimations of the $L^{p(\cdot)}(\mathbb{R}^n)$ norm of the characteristic functions of cubes and the other one about the vector valued boundedness of the fractional maximal operator.

In Section 3 we obtain the $H^{p(\cdot)}(\mathbb{R}^n) - L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the operator T_α corresponding to the case $m > 1$, where $p(\cdot)$ is an exponent function satisfying the log-Hölder continuity conditions (2) and (4), such that $p(A_i x) = p(x)$, $x \in \mathbb{R}^n$, $1 \leq i \leq m$, $0 < p_- \leq p_+ < \frac{n}{\alpha}$ and $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$.

In Section 4 we get the $H^{p(\cdot)}(\mathbb{R}^n) - H^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the Riesz potential I_α where $p(\cdot)$ satisfies (2), (4), $0 < p_- \leq p_+ < \frac{n}{\alpha}$ and $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$.

Notation The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some constant c . The symbol $A \sim B$ stands for $B \lesssim A \lesssim B$. We denote by $Q(z, r)$ the cube centered at $z = (z_1, \dots, z_n)$ with side length r . Given a cube $Q = Q(z, r)$, we set $kQ = Q(z, kr)$ and $l(Q) = r$. For a measurable subset $E \subseteq \mathbb{R}^n$ we denote by $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E respectively. For a function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ we define $\underline{p} = \min(p_-, 1)$; also given a cube Q define $p_-(Q) = \inf\{p(x) : x \in Q\}$ and $p_+(Q) = \sup\{p(x) : x \in Q\}$. As usual we denote with $S(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions and with $S'(\mathbb{R}^n)$ the dual space. If β is the multiindex $\beta = (\beta_1, \dots, \beta_n)$ then $|\beta| = \beta_1 + \dots + \beta_n$.

2. PRELIMINARIES

Given a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ such that $0 < p_- \leq p_+ < \infty$, in the paper [9] E. Nakai and Y. Sawano give a variety of distinct approaches, based on differing definitions, all lead to the same notion of the Hardy space $H^{p(\cdot)}$.

Definition 1. Define $\mathcal{F}_N = \left\{ \varphi \in S'(\mathbb{R}^n) : \sum_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\beta \varphi(x)| \leq 1 \right\}$.

Let $f \in S'(\mathbb{R}^n)$. Denote by \mathcal{M} the grand maximal operator given by

$$\mathcal{M}f(x) = \sup_{t > 0} \sup_{\varphi \in \mathcal{F}_N} |(t^{-n} \varphi(t^{-1} \cdot) * f)(x)|,$$

where N is a large and fixed integer. The variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ is the set of all $f \in S'(\mathbb{R}^n)$ for which $\|\mathcal{M}f\|_{p(\cdot)} < \infty$. In this case we define $\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}f\|_{p(\cdot)}$.

Definition 2. $((p(\cdot), p_0, d) - \text{atom})$. Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$, $0 < p_- \leq p_+ < p_0 \leq \infty$ and $p_0 \geq 1$. Fix an integer $d \geq d_{p(\cdot)} = \min \{l \in \mathbb{N} \cup \{0\} : p_-(n + l + 1) > n\}$. A function a on \mathbb{R}^n is called a $(p(\cdot), p_0, d)$ -atom if there exists a cube Q such that

$a_1)$ $\text{supp}(a) \subset Q$,

$a_2)$ $\|a\|_{p_0} \leq \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{p(\cdot)}}$,

$a_3)$ $\int a(x) x^\alpha dx = 0$ for all $|\alpha| \leq d$.

Definition 3. For sequences of nonnegative numbers $\{\lambda_j\}_{j=1}^\infty$ and cubes $\{Q_j\}_{j=1}^\infty$ and for a function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$, we define

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)) = \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{p(\cdot)}.$$

The space $H_{atom}^{p(\cdot), p_0, d}(\mathbb{R}^n)$ is the set of all distributions $f \in S'(\mathbb{R}^n)$ such that it can be written as

$$(5) \quad f = \sum_{j=1}^\infty \lambda_j a_j$$

in $S'(\mathbb{R}^n)$, where $\{\lambda_j\}_{j=1}^\infty$ is a sequence of non negative numbers, the a_j 's are $(p(\cdot), p_0, d)$ -atoms and $\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)) < \infty$. One defines

$$\|f\|_{H_{atom}^{p(\cdot), p_0, d}} = \inf \mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot))$$

where the infimum is taken over all admissible expressions as in (5).

Theorem 4.6 in [9] asserts that $\|f\|_{H_{atom}^{p(\cdot), p_0, d}} \sim \|f\|_{H^{p(\cdot)}}$, thus we will study the behavior of the operators T_α on atoms.

The following lemmas are crucial to get the principal results.

Lemma 4. (Lemma 2.2. in [9]) Suppose that $p(\cdot)$ is a function satisfying (2), (4) and $0 < p_- \leq p_+ < \infty$.

1) For all cubes $Q = Q(z, r)$ with $z \in \mathbb{R}^n$ and $r \leq 1$, we have

$$|Q|^{\frac{1}{p_-(Q)}} \lesssim |Q|^{\frac{1}{p_+(Q)}}.$$

In particular, we have

$$|Q|^{\frac{1}{p_-(Q)}} \sim |Q|^{\frac{1}{p_+(Q)}} \sim |Q|^{\frac{1}{p(z)}} \sim \|\chi_Q\|_{L^{p(\cdot)}}.$$

2) For all cubes $Q = Q(z, r)$ with $z \in \mathbb{R}^n$ and $r \geq 1$, we have

$$|Q|^{\frac{1}{p_\infty}} \sim \|\chi_Q\|_{L^{p(\cdot)}}.$$

Here the implicit constants in \sim do not depend on z and $r > 0$.

Lemma 5. *Let A be an $n \times n$ invertible matrix. Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a function satisfying (2), (4), $0 < p_- \leq p_+ < \infty$ and $p(Ax) = p(x)$ for all $x \in \mathbb{R}^n$. Given a sequence of cubes $Q_j = Q(z_j, r_j)$, we set $Q_j^* = Q(Az_j, 4Dr_j)$ for each $j \in \mathbb{N}$, where $D = \|A\|$. Then*

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty, p(\cdot)\right) \lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right),$$

for all sequences of nonnegative numbers $\{\lambda_j\}_{j=1}^\infty$ and cubes $\{Q_j\}_{j=1}^\infty$.

Proof. Since $p(Ax) = p(x)$ for all $x \in \mathbb{R}^n$, a change of variable gives

$$\begin{aligned} \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j^*\}_{j=1}^\infty, p(\cdot)\right) &= \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j \chi_{Q_j^*}}{\|\chi_{Q_j^*}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{p(\cdot)} \\ &= \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j \chi_{A^{-1}Q_j^*}}{\|\chi_{A^{-1}Q_j^*}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{p(\cdot)} =: \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{A^{-1}Q_j^*\}_{j=1}^\infty, p(\cdot)\right), \end{aligned}$$

it is easy to check that $Q_j \subset A^{-1}Q_j^*$ for all j . Moreover, there exists a positive universal constant c such that $|A^{-1}Q_j^*| \leq c|Q_j|$ for all j . The same argument utilized in the proof of Lemma 4.8 in [9] works in this case, so

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{A^{-1}Q_j^*\}_{j=1}^\infty, p(\cdot)\right) \lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right).$$

The proof is therefore concluded. \square

Given $0 < \alpha < n$, we define the fractional maximal operator M_α by

$$M_\alpha f(x) = \sup_Q \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

where f is a locally integrable function and the supremum is taken over all the cubes Q which contain x . In the case $\alpha = 0$, the fractional maximal operator reduces to the Hardy-Littlewood maximal operator.

Lemma 6. *Let $0 \leq \alpha < n$, let $p(\cdot) : \mathbb{R}^n \rightarrow (1, \infty)$ such that p satisfies (2), (4) and $1 < p_- \leq p_+ < \frac{n}{\alpha}$. Then for $\theta \in (1, \infty)$ and $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ we have*

$$\left\| \left(\sum_{j=1}^\infty (M_\alpha f_j)^\theta \right)^{\frac{1}{\theta}} \right\|_{q(\cdot)} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{p(\cdot)},$$

for all sequences of bounded measurable functions with compact support $\{f_j\}_{j=1}^\infty$.

Proof. For the case $0 < \alpha < n$, the boundedness of the fractional maximal operator M_α from $L^p(w^p)$ into $L^q(w^q)$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and for all weights w in the Muckenhoupt class $A_{p,q}$ (see [8]) gives the inequality 3.16, Theorem 3.23 in [3], for the pair $(M_\alpha f, f)$, $f \in L^p(\mathbb{R}^n)$. So for $1 < \theta < \infty$,

$$\left\| \left\{ \sum_{j=1}^\infty (M_\alpha f_j)^\theta \right\}^{\frac{1}{\theta}} \right\|_{L^q(w^q)} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{L^p(w^p)}$$

for all weights w in the Muckenhoupt class $A_{p,q}$. Now $w \in A_1$ implies $w^{\frac{1}{q}} \in A_{p,q}$ so

$$\left\| \left\{ \sum_{j=1}^{\infty} (M_{\alpha} f_j)^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{L^q(w)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^{\theta} \right)^{\frac{1}{\theta}} \right\|_{L^p\left(w^{\frac{p}{q}}\right)}$$

for all $w \in A_1$, thus the lemma follows, in this case, from Lemma 4.30 in [3]. For the case $\alpha = 0$, Theorem 4.25 in [3] applies. \square

Lemma 7. *Let $0 \leq \alpha < n$, let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ such that p satisfies (2), (4) and $0 < p_- \leq p_+ < \frac{n}{\alpha}$. If $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, then*

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, q(\cdot)\right) \lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, p(\cdot)\right),$$

for all sequences of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and cubes $\{Q_j\}_{j=1}^{\infty}$.

Proof. Since $l^p \hookrightarrow l^q$, we have

$$\begin{aligned} \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, q(\cdot)\right) &= \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{q(\cdot)}} \right)^q \right\}^{\frac{1}{q}} \right\|_{q(\cdot)} \\ &\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{q(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{q(\cdot)}. \end{aligned}$$

Now from Lemma 4 we obtain $\|\chi_{Q_j}\|_{q(\cdot)} \sim \|\chi_{Q_j}\|_{p(\cdot)} |Q_j|^{-\frac{\alpha}{n}}$. Moreover a simple computation gives

$$|Q_j|^{\frac{\alpha}{n}} \chi_{Q_j}(x) \leq M_{\frac{\alpha p}{2}}(\chi_{Q_j})^{\frac{2}{p}}(x),$$

so

$$\begin{aligned} &\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{Q_j} |Q_j|^{\frac{\alpha}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{q(\cdot)} \lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j M_{\frac{\alpha p}{2}}(\chi_{Q_j})^{\frac{2}{p}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{q(\cdot)} \\ &= \left\| \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j^p M_{\frac{\alpha p}{2}}^p(\chi_{Q_j})^2}{\|\chi_{Q_j}\|_{p(\cdot)}^p} \right\}^{\frac{1}{p}} \right\|_{q(\cdot)} = \left\| \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j^p M_{\frac{\alpha p}{2}}^p(\chi_{Q_j})^2}{\|\chi_{Q_j}\|_{p(\cdot)}^p} \right\}^{\frac{1}{2}} \right\|_{\frac{2q(\cdot)}{p}}^{\frac{2}{p}} \\ &\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \frac{\lambda_j^p \chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}^p} \right\}^{\frac{1}{2}} \right\|_{\frac{2p(\cdot)}{p}}^{\frac{2}{p}} = \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{p(\cdot)} \\ &= \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, p(\cdot)\right), \end{aligned}$$

where the third inequality follows from Lemma 6. \square

3. THE MAIN RESULT

Theorem 8. *Let $m > 1$, let A_1, \dots, A_m be $n \times n$ invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, let $0 \leq \alpha < n$ and let T_α be the integral operator defined by (1). Suppose $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ satisfies (2), (4), $0 < p_- \leq p_+ < \frac{n}{\alpha}$ and $p(A_i x) \equiv p(x)$, $1 \leq i \leq m$. If $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ then T_α can be extended to an $H^{p(\cdot)}(\mathbb{R}^n) - L^{q(\cdot)}(\mathbb{R}^n)$ bounded operator.*

Proof. Let $\max\{1, p_+\} < p_0 < \frac{n}{\alpha}$. Given $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$, from Theorem 4.6 in [9] we have that there exist a sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^\infty$, a sequence of cubes $Q_j = Q(z_j, r_j)$ centered at z_j with side length r_j and $(p(\cdot), p_0, d)$ atoms a_j supported on Q_j , satisfying

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)) \leq c \|f\|_{H^{p(\cdot)}},$$

such that f can be decomposed as $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$, where the convergence is in $H^{p(\cdot)}$ and in L^{p_0} (for the converge in L^{p_0} see Theorem 5 in [10], the same argument works for $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$). We will study the behavior of T_α on atoms. Define $D = \max_{1 \leq i \leq m, \|x\| \leq 1} \{\|A_i(x)\|\}$. Fix $j \in \mathbb{N}$, let a_j be an $(p(\cdot), p_0, d)$ -atom supported on a cube $Q_j = Q(z_j, r_j)$, for each $1 \leq i \leq m$ let $Q_{ji}^* = Q(A_i z_j, 4Dr_j)$. Proposition 1 in [11] gives that T_α is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ for $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, thus

$$\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \lesssim \|a_j\|_{p_0} \lesssim \frac{|Q_j|^{\frac{1}{p_0}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \lesssim \frac{|Q_{ji}^*|^{\frac{1}{q_0}}}{\|\chi_{Q_{ji}^*}\|_{q(\cdot)}},$$

where the last inequality follows from lemma 4. So if $\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \neq 0$ we get

$$(6) \quad 1 \lesssim \frac{|Q_{ji}^*|^{\frac{1}{q_0}}}{\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \|\chi_{Q_{ji}^*}\|_{q(\cdot)}}.$$

We denote $k(x, y) = |x - A_1 y|^{-\alpha_1} \dots |x - A_m y|^{-\alpha_m}$. In view of the moment condition of a_j we have

$$(7) \quad T_\alpha a_j(x) = \int_{Q_j} k(x, y) a_j(y) dy = \int_{Q_j} (k(x, y) - q_{d,j}(x, y)) a_j(y) dy,$$

where $q_{d,j}$ is the degree d Taylor polynomial of the function $y \rightarrow k(x, y)$ expanded around z_j . By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and z_j such that

$$\begin{aligned} |k(x, y) - q_{d,j}(x, y)| &\lesssim |y - z_j|^{d+1} \sum_{k_1 + \dots + k_n = d+1} \left| \frac{\partial^{d+1}}{\partial y_1^{k_1} \dots \partial y_n^{k_n}} k(x, \xi) \right| \\ &\lesssim |y - z_j|^{d+1} \left(\prod_{i=1}^m |x - A_i \xi|^{-\alpha_i} \right) \left(\sum_{l=1}^m |x - A_l \xi|^{-1} \right)^{d+1}. \end{aligned}$$

Now we decompose $\mathbb{R}^n = \bigcup_{i=1}^m Q_{ji}^* \cup R_j$, where $R_j = (\bigcup_{i=1}^m Q_{ji}^*)^c$, at the same time we decompose $R_j = \bigcup_{k=1}^m R_{jk}$ with

$$R_{jk} = \{x \in R_j : |x - A_k z_j| \leq |x - A_i z_j| \text{ for all } i \neq k\}.$$

If $x \in R_j$ then $|x - A_i z_j| \geq 2Dr_j$, since $\xi \in Q_j$ it follows that $|A_i z_j - A_i \xi| \leq Dr_j \leq \frac{1}{2}|x - A_i z_j|$ so

$$|x - A_i \xi| = |x - A_i z_j + A_i z_j - A_i \xi| \geq |x - A_i z_j| - |A_i z_j - A_i \xi| \geq \frac{1}{2}|x - A_i z_j|.$$

If $x \in R_j$, then $x \in R_{jk}$ for some k and since $\alpha_1 + \dots + \alpha_m = n - \alpha$ we obtain

$$\begin{aligned} |k(x, y) - q_{d,j}(x, y)| &\lesssim |y - z_j|^{d+1} \left(\prod_{i=1}^m |x - A_i z_j|^{-\alpha_i} \right) \left(\sum_{l=1}^m |x - A_l z_j|^{-1} \right)^{d+1} \\ &\lesssim r_j^{d+1} |x - A_k z_j|^{-n+\alpha-d-1}, \end{aligned}$$

this inequality allow us to conclude that

$$\begin{aligned} |T_\alpha a_j(x)| &\lesssim \|a_j\|_1 r_j^{d+1} |x - A_k z_j|^{-n+\alpha-d-1} \\ &\lesssim |Q_j|^{1-\frac{1}{p_0}} \|a_j\|_{p_0} r_j^{d+1} |x - A_k z_j|^{-n+\alpha-d-1} \\ &\lesssim \frac{r_j^{n+d+1}}{\|\chi_{Q_j}\|_{p(\cdot)}} |x - A_k z_j|^{-n+\alpha-d-1} \\ &\lesssim \frac{\left(M_{\frac{\alpha n}{n+d+1}}(\chi_{Q_j})(A_k^{-1}x) \right)^{\frac{n+d+1}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}}, \quad \text{if } x \in R_{jk}. \end{aligned}$$

Since $f = \sum_{j=1}^\infty \lambda_j a_j$ in L^{p_0} and T_α is an $L^{p_0} - L^{\frac{np_0}{n-\alpha p_0}}$ bounded operator, we have that $|T_\alpha f(x)| \leq \sum_{j=1}^\infty \lambda_j |T_\alpha a_j(x)|$. So

$$\begin{aligned} |T_\alpha f(x)| &\leq \sum_{j=1}^\infty \lambda_j |T_\alpha a_j(x)| = \sum_{j=1}^\infty \left(\chi_{\bigcup_{i=1}^m Q_{ji}^*}(x) + \chi_{R_j}(x) \right) \lambda_j |T_\alpha a_j(x)| \\ &\lesssim \sum_{j=1}^\infty \sum_{i=1}^m \chi_{Q_{ji}^*}(x) \lambda_j |T_\alpha a_j(x)| + \sum_{j=1}^\infty \sum_{k=1}^m \chi_{R_{jk}}(x) \lambda_j |T_\alpha a_j(x)| \\ &\lesssim \sum_{j=1}^\infty \sum_{i=1}^m \chi_{Q_{ji}^*}(x) \lambda_j |T_\alpha a_j(x)| \\ &\quad + \sum_{j=1}^\infty \sum_{k=1}^m \chi_{R_{jk}}(x) \lambda_j \frac{\left(M_{\frac{\alpha n}{n+d+1}}(\chi_{Q_j})(A_k^{-1}x) \right)^{\frac{n+d+1}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}} = I + II \end{aligned}$$

To study I , if $\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \neq 0$, we apply (6) to obtain, since $\underline{q} \leq 1$,

$$\begin{aligned} \|I\|_{q(\cdot)} &\lesssim \sum_{i=1}^m \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_{ji}^*} |T_\alpha a_j| \right\|_{q(\cdot)} \lesssim \sum_{i=1}^m \left\| \sum_{j=1}^\infty \frac{\lambda_j \chi_{Q_{ji}^*} |T_\alpha a_j| |Q_{ji}^*|^{\frac{1}{q_0}}}{\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \|\chi_{Q_{ji}^*}\|_{q(\cdot)}} \right\|_{q(\cdot)} \\ &\lesssim \sum_{i=1}^m \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j \chi_{Q_{ji}^*} |T_\alpha a_j| |Q_{ji}^*|^{\frac{1}{q_0}}}{\|T_\alpha a_j\|_{L^{q_0}(Q_{ji}^*)} \|\chi_{Q_{ji}^*}\|_{q(\cdot)}} \right)^{\underline{q}} \right\}^{\frac{1}{\underline{q}}} \right\|_{q(\cdot)}, \end{aligned}$$

now we take p_0 near $\frac{n}{\alpha}$ such that $\delta = \frac{1}{q_0}$ satisfies the hypothesis of Lemma 4.11 in [9] to get

$$\lesssim \sum_{i=1}^m \mathcal{A} \left(\{\lambda_j\}_{j=1}^\infty, \{Q_{ji}^*\}_{j=1}^\infty, q(\cdot) \right)$$

Lemma 5 gives

$$\lesssim m \mathcal{A} \left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, q(\cdot) \right)$$

now from Lemma 7,

$$\lesssim \mathcal{A} \left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot) \right) \lesssim \|f\|_{H^{p(\cdot)}}.$$

To study II , we observe that

$$\begin{aligned}
\|II\|_{q(\cdot)} &= \left\| \sum_{j=1}^{\infty} \sum_{k=1}^m \lambda_j \chi_{R_{j,k}}(\cdot) \frac{\left(M_{\frac{\alpha n}{n+d+1}}(\chi_{Q_j})(A_k^{-1} \cdot) \right)^{\frac{n+d+1}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\|_{q(\cdot)} \\
&\lesssim m \left\| \left\{ \sum_{j=1}^{\infty} \lambda_j \frac{\left(M_{\frac{\alpha n}{n+d+1}}(\chi_{Q_j})(A_k^{-1} \cdot) \right)^{\frac{n+d+1}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\}^{\frac{n}{n+d+1}} \right\|_{\frac{n+d+1}{n} q(A_k \cdot)}^{\frac{n+d+1}{n}} \\
&\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \lambda_j \frac{\chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\}^{\frac{n}{n+d+1}} \right\|_{\frac{n+d+1}{n} p(A_k \cdot)}^{\frac{n+d+1}{n}} = \left\| \sum_{j=1}^{\infty} \lambda_j \frac{\chi_{Q_j}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} \\
&\lesssim \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}, p(\cdot)) \lesssim \|f\|_{H^{p(\cdot)}},
\end{aligned}$$

where the second inequality follows from Lemma 6, since $\frac{n+d+1}{n}q > 1$, the third inequality follows from Remark 4.4 in [9] and the second equality follows since $p(A_k x) \equiv p(x)$. Thus

$$\|T_{\alpha} f\|_{q(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}}$$

for all $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$, so the theorem follows from the density of $H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$ in $H^{p(\cdot)}(\mathbb{R}^n)$. \square

Remark 9. Observe that Theorem 8 still holds for $m = 1$ and $0 < \alpha < n$. In particular, if $A_1 = I$, then the Riesz potential is bounded from $H^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.

Remark 10. Suppose $h : \mathbb{R} \rightarrow (0, \infty)$ that satisfies (2) and (4) on \mathbb{R} and $0 < h_- \leq h_+ < \frac{n}{\alpha}$. Let $p(x) = h(|x|)$ for $x \in \mathbb{R}^n$ and for $m > 1$ let A_1, \dots, A_m be $n \times n$ orthogonal matrices such that $A_i - A_j$ is invertible for $i \neq j$. It is easy to check that (2) and (4) hold for p and also that $0 < p_- \leq p_+ < \frac{n}{\alpha}$ and $p(A_i x) \equiv p(x)$, $1 \leq i \leq m$.

Another non trivial example of exponent functions and invertible matrices satisfying the hypothesis of the theorem is the following:

We consider $m = 2$, $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ that satisfies (2) and (4), $0 < p_- \leq p_+ < \frac{n}{\alpha}$, and then we take $p_e(x) = p(x) + p(-x)$, $A_1 = I$ and $A_2 = -I$.

4. $H^{p(\cdot)}(\mathbb{R}^n) - H^{q(\cdot)}(\mathbb{R}^n)$ BOUNDEDNESS OF THE RIESZ POTENTIAL

For $0 < \alpha < n$, let I_{α} be the fractional integral operator (or Riesz potential) defined by

$$(8) \quad I_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) dy,$$

$f \in L^s(\mathbb{R}^n)$, $1 \leq s < \frac{n}{\alpha}$. A well known result of Sobolev gives the boundedness of I_{α} from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In [1] C. Capone, D. Cruz Uribe and A. Fiorenza extend this result to the case of Lebesgue spaces with variable exponents $L^{p(\cdot)}$. In [12] E. Stein and G. Weiss used the theory of harmonic functions of several variables to prove that these operators are bounded from $H^1(\mathbb{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. In [13], M. Taibleson and G. Weiss obtained the boundedness of the Riesz potential I_{α} from the Hardy spaces $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, for $0 < p < 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. We extend these results to the context of Hardy

spaces with variable exponents. The main tools that we use are Lemma 7 and the molecular decomposition developed in [9].

Definition 11. (*Molecules*) Let $0 < p_- \leq p_+ < p_0 \leq \infty$, $p_0 \geq 1$ and $d \in \mathbb{Z} \cap [d_{p(\cdot)}, \infty)$ be fixed. One says that \mathfrak{M} is a $(p(\cdot), p_0, d)$ molecule centered at a cube Q centered at z if it satisfies the following conditions.

- 1) On $2\sqrt{n}Q$, \mathfrak{M} satisfies the estimate $\|\mathfrak{M}\|_{L^{p_0}(2\sqrt{n}Q)} \leq \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{p(\cdot)}}$.
- 2) Outside $2\sqrt{n}Q$, we have $|\mathfrak{M}(x)| \leq \frac{1}{\|\chi_Q\|_{p(\cdot)}} \left(1 + \frac{|x-z|}{l(Q)}\right)^{-2n-2d-3}$. This condition is called the decay condition.
- 3) If β is a multiindex with $|\beta| \leq d$, then we have

$$\int_{\mathbb{R}^n} x^\beta \mathfrak{M}(x) dx = 0.$$

This condition is called the moment condition.

Theorem 12. Let $0 < \alpha < n$ and let I_α be defined by (8). If $p(\cdot)$ is a measurable function that satisfies (2), (4) and $0 < p_- \leq p_+ < \frac{n}{\alpha}$ and if $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, then I_α can be extended to an $H^{p(\cdot)}(\mathbb{R}^n) - H^{q(\cdot)}(\mathbb{R}^n)$ bounded operator.

Proof. Since $2d_{q(\cdot)} + 2 + \alpha + n \geq d_{p(\cdot)}$, as Theorem 8, given p_0 such that $\max\{1, p_+\} < p_0 < \frac{n}{\alpha}$, we can decompose a distribution $f \in H^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$ as $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j is an $(p(\cdot), p_0, 2d_{q(\cdot)} + 2 + \alpha + n)$ -atom supported on the cube Q_j , where the convergence is in $H^{p(\cdot)}(\mathbb{R}^n)$ and in $L^{p_0}(\mathbb{R}^n)$ so it is enough to show that if a is an $(p(\cdot), p_0, 2d_{q(\cdot)} + 2 + \alpha + n)$ -atom supported on the cube Q centered at z , then $cI_\alpha(a)$ is a $(q(\cdot), q_0, d_{q(\cdot)})$ -molecule centered at a cube Q for some fixed constant $c > 0$ independent of the atom a . Indeed, since $f = \sum_{j=1}^\infty \lambda_j a_j$ in $L^{p_0}(\mathbb{R}^n)$ then $I_\alpha f = \sum_{j=1}^\infty \lambda_j I_\alpha(a_j)$ in $L^{\frac{np_0}{n-\alpha p_0}}(\mathbb{R}^n)$ and thus $I_\alpha f = \sum_{j=1}^\infty \lambda_j I_\alpha(a_j)$ in \mathcal{S}' . Now from Theorem 5.2 in [9] we obtain

$$\|I_\alpha f\|_{H^{q(\cdot)}} \lesssim \mathcal{A}(\{\lambda_j\}, \{Q_j\}, q(\cdot))$$

by Lemma 7 and Theorem 4.6 in [9] we have

$$\lesssim \mathcal{A}(\{\lambda_j\}, \{Q_j\}, p(\cdot)) \lesssim \|f\|_{H^{p(\cdot)}},$$

for all $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$, so the theorem follows from the density of $H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$ in $H^{p(\cdot)}(\mathbb{R}^n)$.

Now we show that there exists $c > 0$ such that $cI_\alpha(a)$ is a $(q(\cdot), q_0, d_{q(\cdot)})$ -molecule

- 1) The Sobolev theorem and Lemma 4 give

$$\|I_\alpha(a)\|_{L^{q_0}(2\sqrt{n}Q)} \lesssim \|a\|_{L^{p_0}} \leq \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{p(\cdot)}} \lesssim \frac{|Q|^{\frac{1}{q_0}}}{\|\chi_Q\|_{q(\cdot)}}.$$

- 2) Denote $d = 2d_{q(\cdot)} + 2 + \alpha + n$. As in the proof of Theorem 8 we get, for x outside

$2\sqrt{n}Q$, that

$$\begin{aligned}
|I_\alpha(a)(x)| &\lesssim \frac{l(Q)^{d+1}}{|x-z|^{n-\alpha+d+1}} \|a\|_1 \lesssim \frac{1}{\|\chi_Q\|_{q(\cdot)}} \left(\frac{l(Q)}{|x-z|} \right)^{n-\alpha+d+1} \\
&= \frac{1}{\|\chi_Q\|_{q(\cdot)}} \left(\frac{l(Q)}{|x-z|} \right)^{2n+2d_{q(\cdot)}+3} \\
&\lesssim \frac{1}{\|\chi_Q\|_{q(\cdot)}} \left(\frac{l(Q)}{l(Q)+|x-z|} \right)^{2n+2d_{q(\cdot)}+3} \\
&= \frac{1}{\|\chi_Q\|_{q(\cdot)}} \left(1 + \frac{|x-z|}{l(Q)} \right)^{-2n-2d_{q(\cdot)}-3},
\end{aligned}$$

where the third inequality follows since x outside $2\sqrt{n}Q$ implies $l(Q) + |x-z| < 2|x-z|$.

3) The moment condition was proved by Taibleson and Weiss in [13]. \square

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